Applications of Differential Calculus

CHAPTER 4

Concepts and Skills

In this chapter, we review how to use derivatives to

- find slopes of curves and equations of tangent lines;
- find a function's maxima, minima, and points of inflections;
- describe where the graph of a function is increasing, decreasing, concave upward, and concave downward;
- analyze motion along a line;
- create local linear approximations;
- and work with related rates.

For BC Calculus students, we also review how to

- find the slope of parametric and polar curves
- and use vectors to analyze motion along parametrically defined curves.

A. SLOPE; CRITICAL POINTS

Slope of a curve

If the derivative of $y = f(x)$ exists at $P(x_1, y_1)$, then the *slope* of the curve at P (which is defined to be the slope of the tangent to the curve at P) is $f'(x_1)$, the derivative of $f(x)$ at $x = x_1$.

Critical point

Any c in the domain of f such that either $f'(c) = 0$ or $f'(c)$ is undefined is called a *critical point* or *critical value* of f. If f has a derivative everywhere, we find the critical points by solving the equation $f'(x) = 0$.

EXAMPLE 1

For $f(x) = 4x^3 - 6x^2 - 8$, what are the critical points?

SOLUTION: $f'(x) = 12x^2 - 12x = 12x(x - 1)$,

which equals zero if x is 0 or 1. Thus, 0 and 1 are critical points.

EXAMPLE 2

Find any critical points of $f(x) = 3x^3 + 2x$.

SOLUTION: $f'(x) = 9x^2 + 2$.

Since $f'(x)$ never equals zero (indeed, it is always positive), f has no critical values.

EXAMPLE 3

Find any critical points of $f(x) = (x-1)^{1/3}$.

SOLUTION: $f'(x) = \dfrac{1}{3(x-1)^{2/3}}$.

Although f' is never zero, $x = 1$ is a critical value of f because f' does not exist at $x = 1$.

AVERAGE AND INSTANTANEOUS RATES OF CHANGE.

Both average and instantaneous rates of change were defined in Chapter 3. If as x varies from a to $a + h$, the function f varies from $f(a)$ to $f(a + h)$, then we know that the difference quotient

$$\frac{f(a+h)-f(a)}{h}$$

is the average rate of change of f over the interval from a to $a + h$.

Thus, the *average velocity* of a moving object over some time interval is the change in distance divided by the change in time, the average rate of growth of a colony of fruit flies over some interval of time is the change in size of the colony divided by the time elapsed, the average rate of change in the profit of a company on some gadget with respect to production is the change in profit divided by the change in the number of gadgets produced.

The (instantaneous) rate of change of f at a, or the derivative of f at a, is the limit of the average rate of change as $h \to 0$:

$$f'(a) = \lim_{h\to 0}\frac{f(a+h)-f(a)}{h}.$$

On the graph of $y = f(x)$, the rate at which the y-coordinate changes with respect to the x-coordinate is $f'(x)$, the slope of the curve. The rate at which $s(t)$, the distance traveled by a particle in t seconds, changes with respect to time is $s'(t)$, the velocity of the particle; the rate at which a manufacturer's profit $P(x)$ changes relative to the production level x is $P'(x)$.

EXAMPLE 4

Let $G = 400(15 - t)^2$ be the number of gallons of water in a cistern t minutes after an outlet pipe is opened. Find the average rate of drainage during the first 5 minutes and the rate at which the water is running out at the end of 5 minutes.

SOLUTION: The average rate of change during the first 5 min equals

$$\frac{G(5)-G(0)}{5} = \frac{400\cdot 100 - 400\cdot 225}{5} = -10{,}000 \text{ gal/min}.$$

The average rate of drainage during the first 5 min is 10,000 gal/min.
The instantaneous rate of change at $t = 5$ is $G'(5)$. Since

$$G'(t) = -800(15 - t),$$

$G'(5) = -800(10) = -8000$ gal/min. Thus the rate of drainage at the end of 5 min is 8000 gal/min.

B. TANGENTS AND NORMALS

Tangent to a curve

The *equation of the tangent* to the curve $y = f(x)$ at point $P(x_1, y_1)$ is

$$y - y_1 = f'(x_1)(x - x_1).$$

The line through P that is perpendicular to the tangent, called the *normal* to the curve at P, has slope $-\frac{1}{f'(x_1)}$. Its equation is

$$y - y_1 = -\frac{1}{f'(x_1)}(x - x_1).$$

If the tangent to a curve is horizontal at a point, then the derivative at the point is 0. If the tangent is vertical at a point, then the derivative does not exist at the point.

TANGENTS TO PARAMETRICALLY DEFINED CURVES.

BC ONLY

If the curve is defined parametrically, say in terms of t (as in Chapter 1, page 77), then we obtain the slope at any point from the parametric equations. We then evaluate the slope and the x- and y-coordinates by replacing t by the value specified in the question (see Example 9, page 162).

EXAMPLE 5

Find the equations of the tangent and normal to the curve of $f(x) = x^3 - 3x^2$ at the point $(1, -2)$.

SOLUTION: Since $f'(x) = 3x^2 - 6x$ and $f'(1) = -3$, the equation of the tangent is

$$y + 2 = -3(x - 1) \qquad \text{or} \qquad y + 3x = 1,$$

and the equation of the normal is

$$y + 2 = \frac{1}{3}(x - 1) \qquad \text{or} \qquad 3y - x = -7.$$

EXAMPLE 6

Find the equation of the tangent to $x^2y - x = y^3 - 8$ at the point where $x = 0$.

SOLUTION: Here we differentiate implicitly to get $\frac{dy}{dx} = \frac{1 - 2xy}{x^2 - 3y^2}$.

Since $y = 2$ when $x = 0$ and the slope at this point is $\frac{1-0}{0-12} = -\frac{1}{12}$, the equation of the tangent is

$$y - 2 = -\frac{1}{12}x \qquad \text{or} \qquad 12y + x = 24.$$

EXAMPLE 7

Find the coordinates of any point on the curve of $y^2 - 4xy = x^2 + 5$ for which the tangent is horizontal.

SOLUTION: Since $\frac{dy}{dx} = \frac{x+2y}{y-2x}$ and the tangent is horizontal when $\frac{dy}{dx} = 0$,

then $x = -2y$. If we substitute this in the equation of the curve, we get

$$y^2 - 4y(-2y) = 4y^2 + 5$$

$$5y^2 = 5.$$

Thus $y = \pm 1$ and $x = \pm 2$. The points, then, are $(2, -1)$ and $(-2, 1)$.

EXAMPLE 8

Find the x-coordinate of any point on the curve of $y = \sin^2(x + 1)$ for which the tangent is parallel to the line $3x - 3y - 5 = 0$.

SOLUTION: Since $\frac{dy}{dx} = 2\sin(x+1)\cos(x+1) = \sin 2(x+1)$ and since the given line has slope 1, we seek x such that $\sin 2(x + 1) = 1$. Then

$$2(x+1) = \frac{\pi}{2} + 2n\pi \qquad (n \text{ an integer})$$

or

$$x + 1 = \frac{\pi}{4} + n\pi \quad \text{and} \quad x = \frac{\pi}{4} + n\pi - 1.$$

BC ONLY

EXAMPLE 9

Find the equation of the tangent to $F(t) = (\cos t,\ 2\sin^2 t)$ at the point where $t = \frac{\pi}{3}$.

SOLUTION: Since $\frac{dx}{dt} = -\sin t$ and $\frac{dy}{dt} = 4\sin t \cos t$, we see that

$$\frac{dy}{dx} = \frac{4\sin t\cos t}{-\sin t} = -4\cos t.$$

At $t = \frac{\pi}{3}$, $x = \frac{1}{2}$, $y = 2\left(\frac{\sqrt{3}}{2}\right)^2 = \frac{3}{2}$, and $\frac{dy}{dx} = -2$. The equation of the tangent is

$$y - \frac{3}{2} = -2\left(x - \frac{1}{2}\right) \qquad \text{or} \qquad 4x + 2y = 5.$$

C. INCREASING AND DECREASING FUNCTIONS

CASE I. FUNCTIONS WITH CONTINUOUS DERIVATIVES.

Increasing
Decreasing

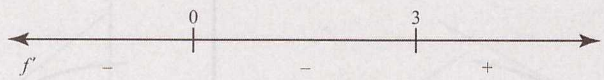
A function $y = f(x)$ is said to be $\begin{matrix} \textit{increasing} \\ \textit{decreasing} \end{matrix}$ on an interval if for all a and b in the interval such that $a < b$, $\begin{matrix} f(b) \geq f(a) \\ f(b) \leq f(a) \end{matrix}$. To find intervals over which $f(x)$ $\begin{matrix} \text{increases} \\ \text{decreases} \end{matrix}$, that is, over which the curve $\begin{matrix} \text{rises} \\ \text{falls} \end{matrix}$, analyze the signs of the derivative to determine where $\begin{matrix} f'(x) \geq 0 \\ f'(x) \leq 0 \end{matrix}$.

EXAMPLE 10

For what values of x is $f(x) = x^4 - 4x^3$, increasing? decreasing?

SOLUTION: $f'(x) = 4x^3 - 12x^2 = 4x^2(x - 3)$.

With critical values at $x = 0$ and $x = 3$, we analyze the signs of f' in three intervals:



The derivative changes sign only at $x = 3$. Thus,

if $x < 3$ $\qquad f'(x) \le 0$ and f is decreasing;

if $x > 3$ $\qquad f'(x) > 0$ and f is increasing.

Note that f is decreasing at $x = 0$ even though $f'(0) = 0$. (See Figure N4–5 on page 168.)

CASE II. FUNCTIONS WHOSE DERIVATIVES HAVE DISCONTINUITIES.

Here we proceed as in Case I, but also consider intervals bounded by any points of discontinuity of f or f'.

EXAMPLE 11

For what values of x is $f(x) = \dfrac{1}{x+1}$ increasing? decreasing?

SOLUTION: $\qquad f'(x) = -\dfrac{1}{(x+1)^2}$.

We note that neither f nor f' is defined at $x = -1$; furthermore, $f'(x)$ never equals zero. We need therefore examine only the signs of $f'(x)$ when $x < -1$ and when $x > -1$.

When $x < -1$, $f'(x) < 0$; when $x > -1$, $f'(x) < 0$. Therefore, f decreases on both intervals. The curve is a hyperbola whose center is at the point $(-1, 0)$.

D. MAXIMUM, MINIMUM, AND INFLECTION POINTS: DEFINITIONS

Local (relative) max/min

The curve of $y = f(x)$ has a *local* (or *relative*) $\begin{matrix}\textit{maximum}\\ \textit{minimum}\end{matrix}$ at a point where $x = c$ if $\begin{matrix} f(c) \geqslant f(x) \\ f(c) \leqslant f(x)\end{matrix}$ for all x in the immediate neighborhood of c. If a curve has a local $\begin{matrix}\text{maximum}\\ \text{minimum}\end{matrix}$ at $x = c$, then the curve changes from $\begin{matrix}\text{rising}\\ \text{falling}\end{matrix}$ to $\begin{matrix}\text{falling}\\ \text{rising}\end{matrix}$ as x increases through c. If a function is differentiable on the closed interval $[a, b]$ and has a local maximum or minimum at $x = c$ $(a < c < b)$, then $f'(c) = 0$. The converse of this statement is not true.

If $f(c)$ is either a local maximum or a local minimum, then $f(c)$ is called a *local extreme value* or *local extremum.* (The plural of *extremum* is *extrema.*)

Global (absolute) max/min

The *global* or *absolute* $\begin{matrix}\textit{maximum}\\ \textit{minimum}\end{matrix}$ of a function on $[a, b]$ occurs at $x = c$ if $\begin{matrix} f(c) \geqslant f(x) \\ f(c) \leqslant f(x)\end{matrix}$ for all x on $[a, b]$.

Concavity

A curve is said to be *concave* $\begin{matrix}\textit{upward}\\ \textit{downward}\end{matrix}$ at a point $P(x_1, y_1)$ if the curve lies $\begin{matrix}\text{above}\\ \text{below}\end{matrix}$ its tangent. If $\begin{matrix} y'' > 0 \\ y'' < 0\end{matrix}$ at P, the curve is concave $\begin{matrix}\text{up}\\ \text{down}\end{matrix}$. In Figure N4–1, the curves sketched in (a) and (b) are concave downward at P while in (c) and (d) they are concave upward at P.

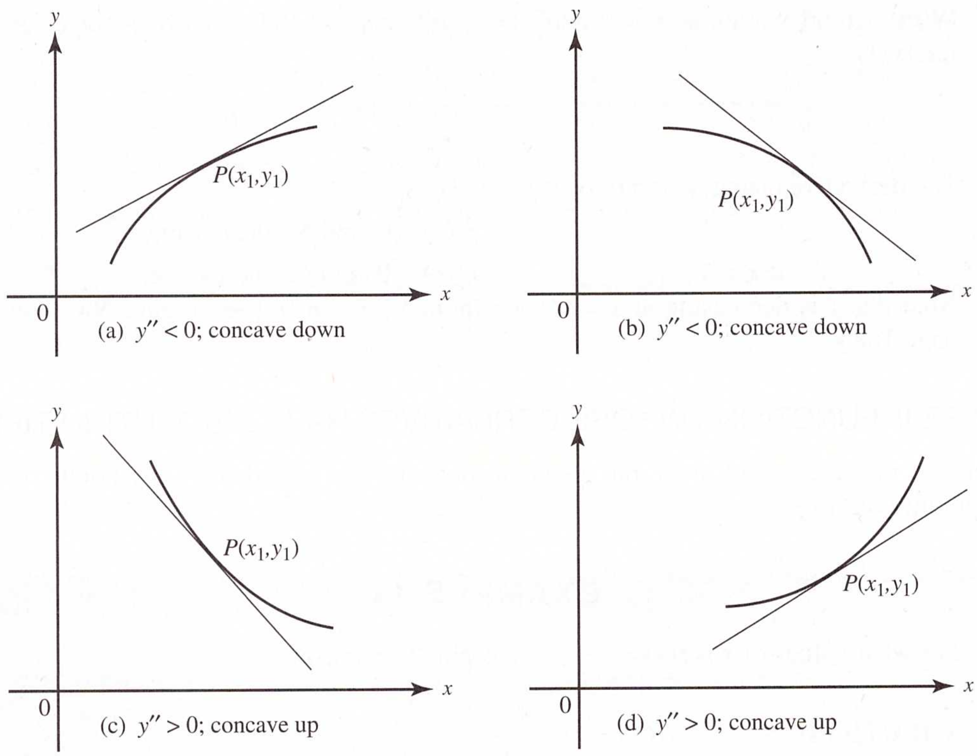


FIGURE N4–1

Point of inflection

A *point of inflection* is a point where the curve changes its concavity from upward to downward or from downward to upward. See §I, page 176, for a table relating a function and its derivatives. It tells how to graph the derivatives of f, given the graph of f. On pages 178 and 262 we graph f, given the graph of f'.

E. MAXIMUM, MINIMUM, AND INFLECTION POINTS: CURVE SKETCHING

CASE I. FUNCTIONS THAT ARE EVERYWHERE DIFFERENTIABLE.

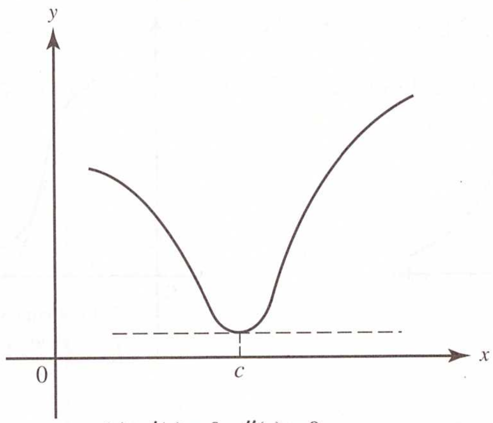
The following procedure is suggested to determine any maximum, minimum, or inflection point of a curve and to sketch the curve.

(1) Find y' and y''.

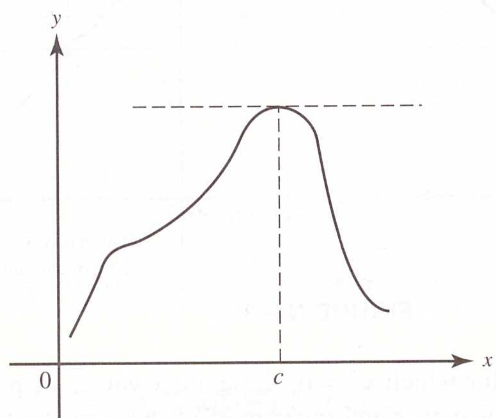
(2) Find all critical points of y, that is, all x for which $y' = 0$. At each of these x's the tangent to the curve is horizontal.

Second Derivative Test

(3) Let c be a number for which y' is 0; investigate the sign of y'' at c. If $y''(c) > 0$, the curve is concave up and c yields a local minimum; if $y''(c) < 0$, the curve is concave down and c yields a local maximum. This procedure is known as the *Second Derivative Test* (for extrema). See Figure N4–2. If $y''(c) = 0$, the Second Derivative Test fails and we must use the test in step (4) below.



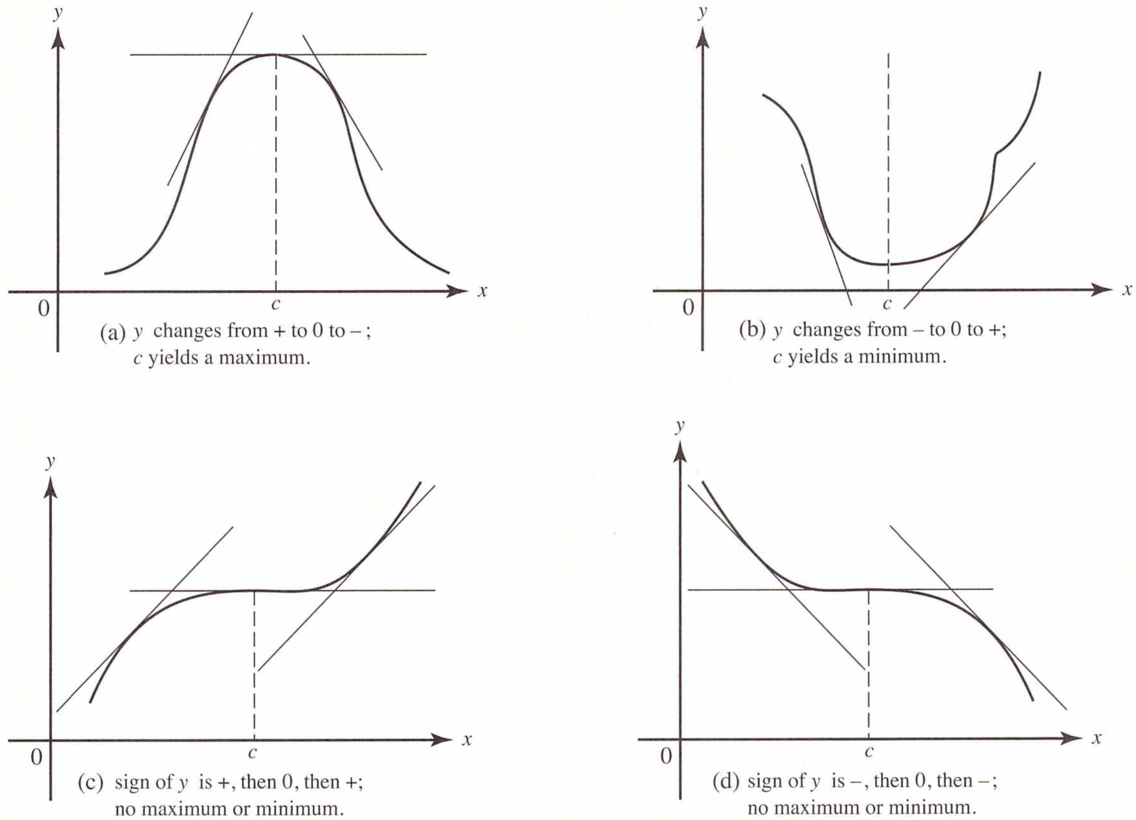
(a) $y'(c) = 0$; $y''(c) > 0$;
c yields a local minimum.



(a) $y'(c) = 0$; $y''(c) < 0$;
c yields a local maximum.

FIGURE N4–2

(4) If $y'(c) = 0$ and $y''(c) = 0$, investigate the signs of y' as x increases through c. If $y'(x) > 0$ for x's (just) less than c but $y'(x) < 0$ for x's (just) greater than c, then the situation is that indicated in Figure N4–3a, where the tangent lines have been sketched as x increases through c; here c yields a local maximum. If the situation is reversed and the sign of y' changes from – to + as x increases through c, then c yields a local minimum. Figure N4–3b shows this case. The schematic sign pattern of y', + 0 – or – 0 +, describes each situation completely. If y' does not change sign as x increases through c, then c yields neither a local maximum nor a local minimum. Two examples of this appear in Figures N4–3c and N4–3d.



(a) y changes from + to 0 to –; c yields a maximum.

(b) y changes from – to 0 to +; c yields a minimum.

(c) sign of y is +, then 0, then +; no maximum or minimum.

(d) sign of y is –, then 0, then –; no maximum or minimum.

FIGURE N4–3

(5) Find all x's for which $y'' = 0$; these are x-values of possible points of inflection. If c is such an x and the sign of y'' changes (from + to – or from – to +) as x increases through c, then c is the x-coordinate of a point of inflection. If the signs do not change, then c does not yield a point of inflection.

The crucial points found as indicated in (1) through (5) above should be plotted along with the intercepts. Care should be exercised to ensure that the tangent to the curve is horizontal whenever $\frac{dy}{dx} = 0$ and that the curve has the proper concavity.

EXAMPLE 12

Find any maximum, minimum, or inflection points on the graph of $f(x) = x^3 - 5x^2 + 3x + 6$, and sketch the curve.

SOLUTION: For the steps listed above:

(1) Here $f'(x) = 3x^2 - 10x + 3$ and $f''(x) = 6x - 10$.

(2) $f'(x) = (3x - 1)(x - 3)$, which is zero when $x = \frac{1}{3}$ or 3.

(3) Since $f'\left(\frac{1}{3}\right) = 0$ and $f''\left(\frac{1}{3}\right) < 0$, we know that the point $\left(\frac{1}{3}, f\left(\frac{1}{3}\right)\right)$ is a local maximum; since $f'(3) = 0$ and $f''(3) > 0$, the point $(3, f(3))$ is a local minimum. Thus, $\left(\frac{1}{3}, \frac{175}{27}\right)$ is a local maximum and $(3, -3)$ a local minimum.

(4) is unnecessary for this problem.

(5) $f''(x) = 0$ when $x = \frac{5}{3}$, and f'' changes from negative to positive as x increases through $\frac{5}{3}$, so the graph of f has an inflection point. See Figure N4–4.

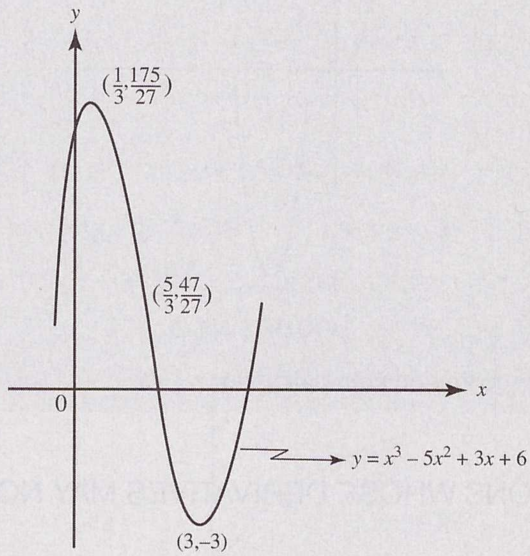


FIGURE N4–4

Verify the graph and information obtained above on your graphing calculator.

EXAMPLE 13

Sketch the graph of $f(x) = x^4 - 4x^3$.

SOLUTION:

(1) $f'(x) = 4x^3 - 12x^2$ and $f''(x) = 12x^2 - 24x$.

(2) $f'(x) = 4x^2(x - 3)$, which is zero when $x = 0$ or $x = 3$.

(3) Since $f''(x) = 12x(x - 2)$ and $f''(3) > 0$ with $f'(3) = 0$, the point $(3, -27)$ is a relative minimum. Since $f''(0) = 0$, the second-derivative test fails to tell us whether $x = 0$ yields a maximum or a minimum.

(4) Since $f'(x)$ does not change sign as x increases through 0, the point $(0, 0)$ yields neither a maximum nor a minimum.

(5) $f''(x) = 0$ when x is 0 or 2; f'' changes signs as x increases through 0 (+ to –), and also as x increases through 2 (– to +). Thus both $(0, 0)$ and $(2, -16)$ are inflection points of the curve.

The curve is sketched in Figure N4–5 on page 168.

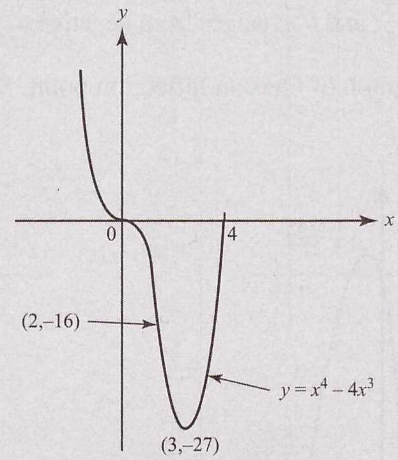


FIGURE N4–5

Verify the preceding on your calculator.

CASE II. FUNCTIONS WHOSE DERIVATIVES MAY NOT EXIST EVERYWHERE.

If there are values of x for which a first or second derivative does not exist, we consider those values separately, recalling that a local maximum or minimum point is one of transition between intervals of rise and fall and that an inflection point is one of transition between intervals of upward and downward concavity.

EXAMPLE 14

Sketch the graph of $y = x^{2/3}$.

SOLUTION: $\frac{dy}{dx} = \frac{2}{3x^{1/3}}$ and $\frac{d^2y}{dx^2} = -\frac{2}{9x^{4/3}}$.

Neither derivative is zero anywhere; both derivatives fail to exist when $x = 0$. As x increases through 0, $\frac{dy}{dx}$ changes from – to +; (0, 0) is therefore a minimum. Note that the tangent is vertical at the origin, and that since $\frac{d^2y}{dx^2}$ is negative everywhere except at 0, the curve is everywhere concave down. See Figure N4–6.

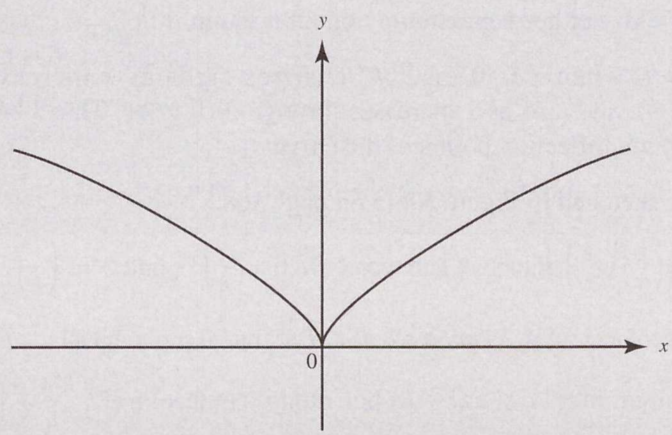


FIGURE N4–6

EXAMPLE 15

Sketch the graph of $y = x^{1/3}$.

SOLUTION: $$\frac{dy}{dx} = \frac{1}{3x^{2/3}} \quad \text{and} \quad \frac{d^2y}{dx^2} = -\frac{2}{9x^{5/3}}.$$

As in Example 14, neither derivative ever equals zero and both fail to exist when $x = 0$. Here, however, as x increases through 0, $\frac{dy}{dx}$ does not change sign. Since $\frac{dy}{dx}$ is positive for all x except 0, the curve rises for all x and can have neither maximum nor minimum points. The tangent is again vertical at the origin. Note here that $\frac{d^2y}{dx^2}$ does change sign (from + to –) as x increases through 0, so that (0, 0) is a point of inflection of the curve. See Figure N4–7.

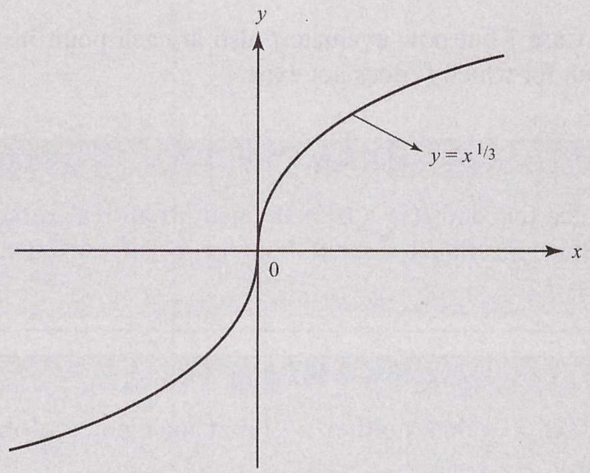


FIGURE N4–7

Verify the graph on your calculator.

F. GLOBAL MAXIMUM OR MINIMUM

CASE I. DIFFERENTIABLE FUNCTIONS.

If a function f is differentiable on a closed interval $a \leqslant x \leqslant b$, then f is also continuous on the closed interval $[a,b]$ and we know from the Extreme Value Theorem (page 101) that f attains both a (global) maximum and a (global) minimum on $[a,b]$. To find these, we solve the equation $f'(x) = 0$ for critical points on the interval $[a,b]$, then evaluate f at each of those and also at $x = a$ and $x = b$. The largest value of f obtained is the global max, and the smallest the global min.

EXAMPLE 16

Find the global max and global min of f on (a) $-2 \leqslant x \leqslant 3$, and (b) $0 \leqslant x \leqslant 3$, if $f(x) = 2x^3 - 3x^2 - 12x$.

SOLUTION:

(a) $f'(x) = 6x^2 - 6x - 12 = 6(x + 1)(x - 2)$, which equals zero if $x = -1$ or 2. Since $f(-2) = -4$, $f(-1) = 7$, $f(2) = -20$, and $f(3) = -9$, the global max of f occurs at $x = -1$ and equals 7, and the global min of f occurs at $x = 2$ and equals -20.

(b) Only the critical value 2 lies in [0,3]. We now evaluate f at 0, 2, and 3. Since $f(0) = 0$, $f(2) = -20$, and $f(3) = -9$, the global max of f equals 0 and the global min equals -20.

CASE II. FUNCTIONS THAT ARE NOT EVERYWHERE DIFFERENTIABLE.

We proceed as for Case I but now evaluate f also at each point in a given interval for which f is defined but for which f' does not exist.

EXAMPLE 17

The absolute-value function $f(x) = |x|$ is defined for all real x, but $f'(x)$ does not exist at $x = 0$. Since $f'(x) = -1$ if $x < 0$, but $f'(x) = 1$ if $x > 0$, we see that f has a global min at $x = 0$.

EXAMPLE 18

The function $f(x) = \frac{1}{x}$ has neither a global max nor a global min on *any* interval that contains zero (see Figure N2–4, page 90). However, it does attain both a global max and a global min on every closed interval that does not contain zero. For instance, on [2,5] the global max of f is $\frac{1}{2}$, the global min $\frac{1}{5}$.

G. FURTHER AIDS IN SKETCHING

It is often very helpful to investigate one or more of the following before sketching the graph of a function or of an equation:

(1) Intercepts. Set $x = 0$ and $y = 0$ to find any y- and x-intercepts respectively.

(2) Symmetry. Let the point (x, y) satisfy an equation. Then its graph is symmetric about
the x-axis if $(x, -y)$ also satisfies the equation;
the y-axis if $(-x, y)$ also satisfies the equation;
the origin if $(-x, -y)$ also satisfies the equation.

(3) Asymptotes. The line $y = b$ is a horizontal asymptote of the graph of a function f if either $\lim_{x\to\infty} f(x) = b$ or $\lim_{x\to-\infty} f(x) = b$. If $f(x) = \frac{P(x)}{Q(x)}$, inspect the degrees of $P(x)$ and $Q(x)$, then use the Rational Function Theorem, page 96. The line $x = c$ is a vertical asymptote of the rational function $\frac{P(x)}{Q(x)}$ if $Q(c) = 0$ but $P(c) \neq 0$.

(4) Points of discontinuity. Identify points not in the domain of a function, particularly where the denominator equals zero.

EXAMPLE 19

Sketch the graph of $y = \frac{2x+1}{x-1}$.

SOLUTION: If $x = 0$, then $y = -1$. Also, $y = 0$ when the numerator equals zero, which is when $x = -\frac{1}{2}$. A check shows that the graph does not possess any of the symmetries described above. Since $y \to 2$ as $x \to \pm\infty$, $y = 2$ is a horizontal asymptote; also, $x = 1$ is a vertical asymptote. The function is defined for all reals except $x = 1$; the latter is the only point of discontinuity.

We find derivatives: $y' = -\frac{3}{(x-1)^2}$ and $y'' = \frac{6}{(x-1)^3}$.

From y' we see that the function decreases everywhere (except at $x = 1$), and from y'' that the curve is concave down if $x < 1$, up if $x > 1$. See Figure N4–8.

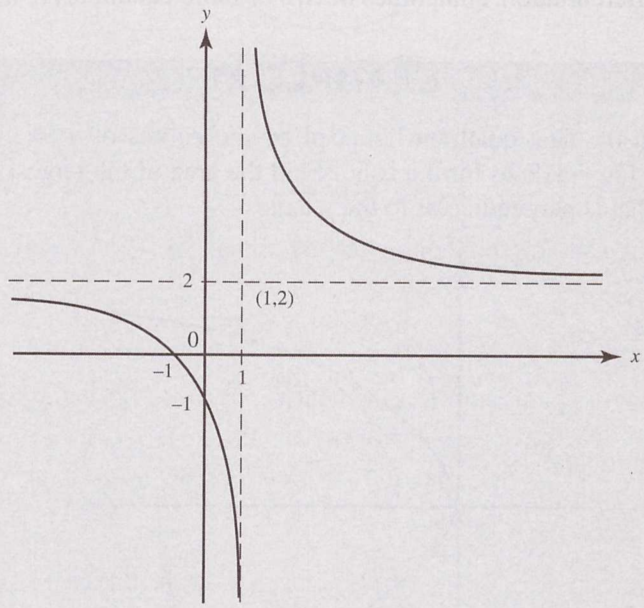


FIGURE N4–8

Verify the preceding on your calculator, using [-4,4] × [–4,8].

EXAMPLE 20

Describe any symmetries of the graphs of
(a) $3y^2 + x = 2$; (b) $y = x + \frac{1}{x}$; (c) $x^2 - 3y^2 = 27$.

SOLUTIONS:

(a) Suppose point (x, y) is on this graph. Then so is point $(x, -y)$, since $3(-y)^2 + x = 2$ is equivalent to $3y^2 + x = 2$. Then (a) is symmetric about the x-axis.

(b) Note that point $(-x, -y)$ satisfies the equation if point (x, y) does:

$$(-y) = (-x) + \frac{1}{(-x)} \leftrightarrow y = x + \frac{1}{x}.$$

Therefore the graph of this function is symmetric about the origin.

(c) This graph is symmetric about the x-axis, the y-axis, and the origin. It is easy to see that, if point (x, y) satisfies the equation, so do points $(x, -y)$, $(-x, y)$, and $(-x, -y)$.

H. OPTIMIZATION: PROBLEMS INVOLVING MAXIMA AND MINIMA

The techniques described above can be applied to problems in which a function is to be maximized (or minimized). Often it helps to draw a figure. If y, the quantity to be maximized (or minimized), can be expressed explicitly in terms of x, then the procedure outlined above can be used. If the domain of y is restricted to some closed interval, one should always check the endpoints of this interval so as not to overlook possible extrema. Often, implicit differentiation, sometimes of two or more equations, is indicated.

EXAMPLE 21

The region in the first quadrant bounded by the curves of $y^2 = x$ and $y = x$ is rotated about the y-axis to form a solid. Find the area of the largest cross section of this solid that is perpendicular to the y-axis.

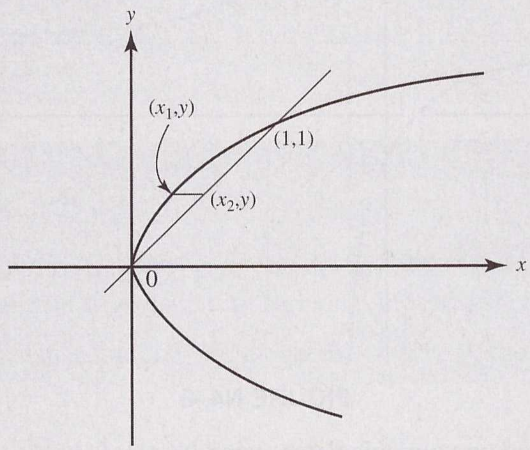


FIGURE N4–9

SOLUTION: See Figure N4–9. The curves intersect at the origin and at (1,1), so $0 < y < 1$. A cross section of the solid is a ring whose area A is the difference between the areas of two circles, one with radius x_2, the other with radius x_1. Thus

$$A = \pi x_2^{\,2} - \pi x_1^{\,2} = \pi(y^2 - y^4); \quad \frac{dA}{dy} = \pi(2y - 4y^3) = 2\pi y(1 - 2y^2).$$

The only relevant zero of the first derivative is $y = \frac{1}{\sqrt{2}}$. There the area A is

$$A = \pi\left(\frac{1}{2} - \frac{1}{4}\right) = \frac{\pi}{4}.$$

Note that $\frac{d^2A}{dy^2} = \pi(2 - 12y^2)$ and that this is negative when $y = \frac{1}{\sqrt{2}}$, assuring a maximum there. Note further that A equals zero at each endpoint of the interval [0,1] so that $\frac{\pi}{4}$ is the global maximum area.

EXAMPLE 22

The volume of a cylinder equals V cubic inches, where V is a constant. Find the proportions of the cylinder that minimize the total surface area.

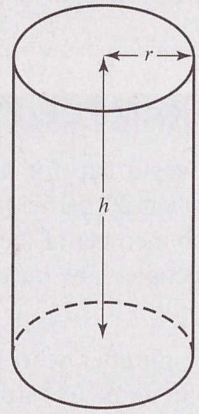


FIGURE N4–10

SOLUTION: We know that the volume is

$$V = \pi r^2 h \tag{1}$$

where r is the radius and h the height. We seek to minimize S, the total surface area, where

$$S = 2\pi r^2 + 2\pi rh \tag{2}$$

Solving (1) for h, we have $h = \dfrac{V}{\pi r^2}$, which we substitute in (2):

$$S = 2\pi r^2 + 2\pi r \frac{V}{\pi r^2} = 2\pi r^2 + \frac{2V}{r}. \tag{3}$$

Differentiating (3) with respect to r yields

$$\frac{dS}{dr} = 4\pi r - \frac{2V}{r^2}.$$

Now we set $\dfrac{dS}{dr}$ equal to zero to determine the conditions that make S a minimum:

$$4\pi r - \frac{2V}{r^2} = 0$$

$$4\pi r = \frac{2V}{r^2}$$

$$4\pi r = \frac{2\left(\pi r^2 h\right)}{r^2}$$

$$2r = h.$$

The total surface area of a cylinder of fixed volume is thus a minimum when its height equals its diameter.

(Note that we need not concern ourselves with the possibility that the value of r that renders $\frac{dS}{dr}$ equal to zero will produce a maximum surface area rather than a minimum one. With V fixed, we can choose r and h so as to make S as large as we like.)

EXAMPLE 23

A charter bus company advertises a trip for a group as follows: At least 20 people must sign up. The cost when 20 participate is \$80 per person. The price will drop by \$2 per ticket for each member of the traveling group in excess of 20. If the bus can accommodate 28 people, how many participants will maximize the company's revenue?

SOLUTION: Let x denote the number who sign up in excess of 20. Then $0 \leqq x \leqq 8$. The total number who agree to participate is $(20 + x)$, and the price per ticket is $(80 - 2x)$ dollars. Then the revenue R, in dollars, is

$$\begin{aligned} R &= (20 + x)(80 - 2x), \\ R'(x) &= (20 + x)(-2) + (80 - 2x) \cdot 1 \\ &= 40 - 4x. \end{aligned}$$

$R'(x)$ is zero if $x = 10$. Although $x = 10$ yields maximum R—note that $R''(x) = -4$ and is always negative—this value of x is not within the restricted interval. We therefore evaluate R at the endpoints 0 and 8: $R(0) = 1600$ and $R(8) = 28 \cdot 64 = 1792$, 28 participants will maximize revenue.

EXAMPLE 24

A utilities company wants to deliver gas from a source S to a plant P located across a straight river 3 miles wide, then downstream 5 miles, as shown in Figure N4–11. It costs \$4 per foot to lay the pipe in the river but only \$2 per foot to lay it on land.

(a) Express the cost of laying the pipe in terms of u.

(b) How can the pipe be laid most economically?

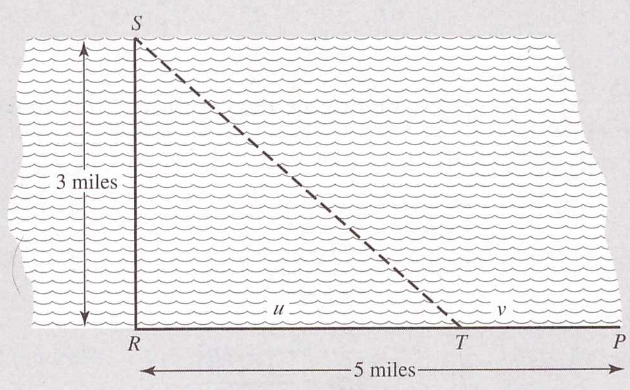


FIGURE N4–11

SOLUTIONS:

(a) Note that the problem "allows" us to (1) lay all of the pipe in the river, along a line from S to P; (2) lay pipe along SR, in the river, then along RP on land; or (3) lay some pipe in the river, say, along ST, and lay the rest on land along TP. When T coincides with P, we have case (1), with $v = 0$; when T coincides with R, we have case (2), with $u = 0$. Case (3) includes both (1) and (2).

In any event, we need to find the lengths of pipe needed (that is, the distances involved); then we must figure out the cost.

In terms of u:

	In the River	On Land
Distances:		
miles	$ST = \sqrt{9+u^2}$	$TP = v = 5 - u$
feet	$ST = 5280\sqrt{9+u^2}$	$TP = 5280(5-u)$
Costs (dollars):	$4(5280)\sqrt{9+u^2}$	$2[5280(5-u)]$

If $C(u)$ is the total cost,

$$\begin{aligned} C(u) &= 21{,}120\sqrt{9+u^2} + 10{,}560(5-u) \\ &= 10{,}560(2\sqrt{9+u^2} + 5 - u). \end{aligned}$$

(b) We now minimize $C(u)$:

$$C'(u) = 10{,}560\left(2 \cdot \frac{1}{2}\frac{2u}{\sqrt{9+u^2}} - 1\right) = 10{,}560\left(\frac{2u}{\sqrt{9+u^2}} - 1\right).$$

We now set $C'(u)$ equal to zero and solve for u:

$$\frac{2u}{\sqrt{9+u^2}} - 1 = 0 \rightarrow \frac{2u}{\sqrt{9+u^2}} = 1 \rightarrow \frac{4u^2}{9+u^2} = 1,$$

where, in the last step, we squared both sides; then

$$4u^2 = 9 + u^2, \qquad 3u^2 = 9, \qquad u^2 = 3, \qquad u = \sqrt{3},$$

where we discard $u = -\sqrt{3}$ as meaningless for this problem.
The domain of $C(u)$ is $[0,5]$ and C is continuous on $[0,5]$. Since

$$\begin{aligned} C(0) &= 10{,}560\left(2\sqrt{9}+5\right) = \$116{,}160, \\ C(5) &= 10{,}560\left(2\sqrt{34}\right) \approx \$123{,}150, \\ C\left(\sqrt{3}\right) &= 10{,}560\left(2\sqrt{12}+5-\sqrt{3}\right) = \$107{,}671, \end{aligned}$$

So $u = \sqrt{3}$ yields minimum cost. Thus, the pipe can be laid most economically if some of it is laid in the river from the source S to a point T that is $\sqrt{3}$ miles toward the plant P from R, and the rest is laid along the road from T to P.